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Bi-level Multi-objective Programming Problems with Fuzzy Parameters: Modified TOPSIS Approach

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Abstract: In this paper, a modified TOPSIS (techniques for order preference by similarity to ideal solution) approach for solving bi-level multi-objective programming (BL-MOP) problems with fuzzy parameters is presented. These fuzzy parameters are assumed to be characterized by fuzzy numerical data, reflecting the experts' imprecise or fuzzy understanding of the nature of the parameters in the problem formulation process. Firstly, the corresponding non-fuzzy bi-level programming model is introduced based on the α-level set. Secondly, a modified TOPSIS approach is developed, in which the fuzzy goal programming (FGP) approach is used to solve the conflicting bi-objective distance functions instead of max-min operator. As the FGP approach utilized to achieve the highest degree of each membership goal by minimizing the sum of the unwanted deviational variables. Finally, an algorithm to clarify the modified TOPSIS approach, as well as Illustrative numerical example and comparison with the existing methods, are presented.

Keywords: Bi-level Programming, Fuzzy Sets, Fuzzy Parameters, TOPSIS, Fuzzy Goal Programming, Multi-objective Programming

1. Introduction

Bi-level mathematical programming (BLMP) is defined as mathematical programming that solves decentralized planning problems with two decision makers (DMs) in two-levels or hierarchical organization [5, 15, 17]. The basic concept of BLMP is that the upper-level decision maker (ULDM) (the leader) sets his goals and/or decisions and then asks each subordinate level of the organization for their optima which are calculated in isolation; the lower-level decision maker (LLDM) (the follower) decisions are then submitted and modified by the ULDM with the consideration of the overall benefit of the organization; the process is continued until a satisfactory solution is reached [6, 7].

Techniques for order performance by similarity to ideal solution (TOPSIS), one of the known classical multiple criteria decision-making (MCDM) method, bases upon the concept that the chosen alternative should have the shortest distance from the positive ideal solution (PIS) and the farthest distance from the negative ideal solution (NIS). It was first developed by Hwang and Yoon [11] for solving a multiple attribute decision-making problem. A similar concept has also been pointed out by Zeleny [24]. Lai et al. [12] extended the concept of TOPSIS to develop a methodology for solving multiple objective decision-making (MODM) problems. Also, Abo-Sinna [2] proposed TOPSIS approach to solve multi-objective dynamic programming (MODP) problems. As Abo-Sinna showed that using the fuzzy max–min operator with non-linear membership functions, the obtained solutions are always non-dominated by the original MODP problems.

Further extensions of TOPSIS for large scale multi-objective non-linear programming problems with block angular structure was presented by Abo-Sinna et al. in [1, 3]. Deng et al. [10] formulated the inter-company comparison process as a multi-criteria analysis model, and presented an effective approach by modifying TOPSIS for solving such a problem. Chen [9] extended the concept of TOPSIS to develop a methodology for solving multi-person multi-

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criteria decision-making problems in a fuzzy environment. Recently, Baky and Abo-Sinna [4] extended the TOPSIS approach for solving bi-level multi-objective decision-making problem. Baky et al. proposed the TOPSIS approach for bi-level multi-objective programming problems with fuzzy parameters [8].

Generally, TOPSIS provides a broader principle of compromise for solving multiple criteria decision-making problems. It transfers m-objectives, which are conflicting and non-commensurable, into two objectives (the shortest distance from the PIS and the longest distance from the NIS). They are commensurable and most of time conflicting. Then, the bi-objective problem can be solved by using membership functions of fuzzy set theory to represent the satisfaction level for both criteria and obtain TOPSIS's compromise solution. The max–min operator is then considered as a suitable one to solve the confliction between the new criteria [1, 3, 11].

It has been observed that, in most real-world situations, for example, power markets and business management, the possible values of the parameters are often only imprecisely or ambiguously known to the experts and cannot be described by precise values. With this observation, it would be certainly more appropriate to interpret the experts understanding of the parameters as fuzzy numerical data which can be represented by means of fuzzy sets [7, 20, 23]. Therefore, this paper study the bi-level multi-objective programming (BL-MOP) problems with fuzzy parameters.

Fuzzy goal programming (FGP) approach was viewed to be an efficient methodology for solving multi-objective programming problems by Mohamed [14]. Thereafter, the concept of FGP approach was extended by Moitra et al. [22] for obtaining a satisfactory solution for bi-level programming problems. Recently, Baky extended FGP approach for solving a decentralized BL-MOP problem [5] and MLP problem [6]. Thus, we make use of the FGP approach to solve the confliction between the bi-objective distance functions instead of the conventional max-min decision model.

This study presents a modified TOPSIS approach for solving BL-MOP problem with fuzzy parameters. To formulate the non-fuzzy model of the problem for a desired value of α, the uncertain model is converted into a deterministic BL-MOP problem using the α-level set of fuzzy numbers. Then, a modified TOPSIS approach is used to solve the deterministic model. Therefore, the confliction between the bi-objective distance functions is solved by using the FGP approach. Since, to overcome the shortcomings of the classical approaches, a FGP model is used to minimize the group regret of degree of satisfaction for the bi-objective distance functions. In order to achieve the highest degree (unity) of each of the defined membership goal by minimizing their deviational variables and thereby obtaining the most satisfactory solution for both of them. Procedures for the developed modified TOPSIS approach are also introduced.

The remainder of this paper is organized as follows. Section 2 presents some preliminaries. Section 3 and section 4 briefly discuss problem formulation and crisp model formulation of BL-MOP problem with fuzzy parameters. The proposed modified TOPSIS approach and an algorithm for solving BL-MOP problem with fuzzy parameters is developed in section 5. The following section presents an illustrative numerical example to demonstrate the proposed modified TOPSIS approach. The concluding remarks are made in section 7.

2. Preliminaries

This section presents some basic concepts of fuzzy set theory and distance measures, for more details see [21, 23] for fuzzy set theory and [1-3] for distance measures.

Definition 2.1: Let R be the space of real numbers. A Fuzzy set $\tilde{A}$ is a set of ordered pairs $\left\{\left(x, \mu_{\tilde{A}}(x)\right) \middle| x \in R\right\}$, where $\mu_{\tilde{A}}(x): \rightarrow [0,1]$ is called membership function of fuzzy set.

Definition 2.2: A convex fuzzy set, $\tilde{A}$, is a fuzzy set in which: $\forall$ x, y $\in$ R, $\forall \lambda \in$ [0, 1] $\mu_{\tilde{A}}(\lambda x + (1-\lambda) y \geq \min\left[\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\right]$.

Definition 2.3: Triangular fuzzy number (TFN) is a convex fuzzy set which is defined as $\tilde{A} = (x, \mu_{\tilde{A}}(x))$ where:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{(x-a)}{(b-a)} & a \leq x \leq b \\ \frac{(c-x)}{(c-b)} & b \leq x \leq c \\ 0 & other\ wise \end{cases} \quad (1)$$

For convenience, TFN represented by three real parameters (a, b, c) which are $(a \leq b \leq c)$ will be denoted by the triangle a, b, c.

Definition 2.4: The α –level set of a fuzzy set $\tilde{A}$ is a non-fuzzy set denoted by $(\tilde{A})_\alpha$ for which the degree of it's membership functions exceed or equal to a real number $\alpha \in [0, 1]$, i.e. $(\tilde{A})_\alpha = \{x | \mu_{\tilde{A}}(x) \geq \alpha\}$.

The α-level set of $\tilde{a}$ is then; $\tilde{a}_\alpha = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U]$ that is $\tilde{a}_\alpha^L = (1-\alpha)a + \alpha b$, and $\tilde{a}_\alpha^U = (1-\alpha)c + \alpha b$, where, $\tilde{a}_\alpha^L$ and $\tilde{a}_\alpha^U$ represent the lower and upper cuts respectively.

Consider the vector of objective functions $F(x) = \left(f_1(x), f_2(x), \dots, f_m(x)\right)$ to be minimized and ideal vector of objective functions $F^* = (f_1^*, f_2^*, \dots, f_m^*)$ (ideal point- reference point- positive ideal solution (PIS)) in the m-objective space. And consider the vector of anti-ideal solution of objective functions $F^- = (f_1^-, f_2^-, \dots, f_m^-)$ (anti-ideal point – nadir point – negative ideal solution (NIS)). Where $f_j^* = \underbrace{min}_{x \in G} f_j(x)$ and $f_j^- = \underbrace{max}_{x \in G} f_j(x), j = 1,2, \dots, m$. And G is a convex constraints feasible set. As the measure of "closeness", L_p-metric is used. The L_p-metric defines the distance between two points $F(x)$ and F^*. If the objective functions $f_j(x), j = 1,2, \dots, m$, are not expressed in commensurable units, then a scaling function for every objective function, usually, this dimensionless is the interval [0, 1]. In this case, the following metric could be used:

$$d_p = \left\{\sum_{j=1}^{m} w_j^p \left[\frac{f_j^* - f_j(x)}{f_j^* - f_j^-}\right]^p\right\}^{\frac{1}{p}}, p = 1,2,\dots,\infty. \quad (2)$$

where $w_j, j = 1,2,\dots,m$, are the relative importance (weights) of objectives.

3. Problem Formulation

Assume that there are two levels in a hierarchy structure with ULDM and LLDM. Let the vector of decision variables $x = (x_1, x_2) \in R^n$ be partitioned between the two decision makers. The ULDM has control over the vector $x_1 \in R^{n_1}$ and the LLDM has control over the vector $x_2 \in R^{n_2}$, where $n = n_1 + n_2$. Furthermore, assume that

$$\tilde{F}_i(x_1, x_2): R^{n_1} \times R^{n_2} \to R^{m_i}, i = 1,2 \quad (3)$$

are the upper-level and the lower-level vector objective functions, respectively. So the BL-MOP problem with fuzzy parameters of minimization-type may be formulated as follows [4-8, 15]:

[1st Level]

$$\underbrace{min}_{x_1} \tilde{F}_1(x_1, x_2) = \underbrace{min}_{x_1} \left(\tilde{f}_{11}(x_1, x_2), \tilde{f}_{12}(x_1, x_2), \dots, \tilde{f}_{1m_1}(x_1, x_2)\right), \quad (4)$$

$where\ x_2\ solves$

[2nd Level]

$$\underbrace{min}_{x_2} \tilde{F}_2(x_1, x_2) = \underbrace{min}_{x_2} \left(\tilde{f}_{21}(x_1, x_2), \tilde{f}_{22}(x_1, x_2), \dots, \tilde{f}_{2m_2}(x_1, x_2)\right), \quad (5)$$

$subject\ to$

$$x \in G = \left\{x = (x_1, x_2) \in R^n \middle| \tilde{A}_1 x_1 + \tilde{A}_2 x_2 \begin{pmatrix} \le \\ = \\ \ge \end{pmatrix} \tilde{b}, x \ge 0, \tilde{b} \in R^m\right\} \ne \varphi, \quad (6)$$

where

$$\tilde{f}_{ij}(x) = \tilde{c}_{11}^{ij} x_{11} + \tilde{c}_{12}^{ij} x_{12} + \dots + \tilde{c}_{1n_1}^{ij} x_{1n_1} + \tilde{c}_{21}^{ij} x_{21} + \tilde{c}_{22}^{ij} x_{22} + \dots + \tilde{c}_{2n_2}^{ij} x_{2n_2}, \quad (7)$$

and where m_i, $i = 1,2$ are the number of DM_i's objective functions, m is the number of constraints, $\tilde{c}_k^{ij} = \left(\tilde{c}_{k1}^{ij}, \tilde{c}_{k2}^{ij}, \dots, \tilde{c}_{kn_k}^{ij}\right), k = 1,2$ and $\tilde{c}_{kn_k}^{ij}$ are constants, $\tilde{A}_i$ are the coefficient matrices of size $m \times n_i$, $i = 1,2$., the control variables $x_1 = \left(x_1^1, x_1^2, \dots, x_1^{n_1}\right)$ and $x_2 = \left(x_2^1, x_2^2, \dots, x_2^{n_2}\right)$, and G is the bi-level convex constraints feasible choice set.

4. Crisp Model Formulation of BL-MOP Problem with Fuzzy Parameters

The individual optimal solution of ULDM and LLDM objective functions would be considered when scaling every objective function. Then for a prescribed value of α, minimization-type objective function [7, 16, 19, 20], $\tilde{f}_{ij}(x)$ $(i = 1,2)$, $(j = 1,2,\dots,m_i)$ can be replaced by the lower bound of its α-level.

$$\left(\tilde{f}_{ij}(x)\right)_\alpha^L = \left(\tilde{c}_1^{ij}\right)_\alpha^L x_1 + \left(\tilde{c}_2^{ij}\right)_\alpha^L x_2 \ (i = 1,2), (j = 1,2,\dots,m_i) \quad (8)$$

Similarly, maximization-type objective function, $\tilde{f}_{ij}(x)$ $(i = 1,2)$, $(j = 1,2,\dots,m_i)$ can be replaced by the upper bound of its α-level (α-cut)

$$\left(\tilde{f}_{ij}(x)\right)_\alpha^U = \left(\tilde{c}_1^{ij}\right)_\alpha^U x_1 + \left(\tilde{c}_2^{ij}\right)_\alpha^U x_2 \ (i = 1,2), (j = 1,2,\dots,m_i) \quad (9)$$

The inequality constraints,

$$\sum_{j=1}^{n} \tilde{A}_{ij}\, x_j \ge \tilde{b}_i \ (i = 1,2,\dots,r_1) \quad (10)$$

$$\sum_{j=1}^{n} \tilde{A}_{ij}\, x_j \le \tilde{b}_i \ (i = r_1 + 1,\dots,r_2) \quad (11)$$

Can be rewritten by the following constraints as [7, 13, 19, 23]:

$$\sum_{j=1}^{n} \left(\tilde{A}_{ij}\right)_\alpha^U x_j \ge \left(\tilde{b}_i\right)_\alpha^L (i = 1,2,\dots,r_1) \quad (12)$$

$$\sum_{j=1}^{n} \left(\tilde{A}_{ij}\right)_\alpha^L x_j \le \left(\tilde{b}_i\right)_\alpha^U (i = r_1 + 1,\dots,r_2) \quad (13)$$

For equality constraints;

$$\sum_{j=1}^{n} \tilde{A}_{ij}\, x_j = \tilde{b}_i \ (i = r_2 + 1,\dots,m) \quad (14)$$

Can be replaced by two equivalent constraints;

$$\sum_{j=1}^{n} \left(\tilde{A}_{ij}\right)_\alpha^L x_j \le \left(\tilde{b}_i\right)_\alpha^U (i = r_2 + 1,\dots,m) \quad (15)$$

$$\sum_{j=1}^{n} \left(\tilde{A}_{ij}\right)_\alpha^U x_j \ge \left(\tilde{b}_i\right)_\alpha^L (i = r_2 + 1,\dots,m) \quad (16)$$

For proof of equivalency of the above equations (14) and (15)&(16), see Lee and Li [13]. Therefore, for a prescribed value of α, the minimization-type α-(BL-MOP) problem reduced to the following problem:

[1st Level]

$$\underbrace{min}_{x_1}\left(\tilde{F}_1(x)\right)_\alpha^L = \underbrace{min}_{x_1}\left(\left(\tilde{f}_{11}(x)\right)_\alpha^L,\left(\tilde{f}_{12}(x)\right)_\alpha^L,\dots,\left(\tilde{f}_{1m_1}(x)\right)_\alpha^L\right), \quad (17)$$

$where\ x_2\ solves$

[2nd Level]

$$\underbrace{min}_{x_2}\left(\tilde{F}_2(x)\right)_\alpha^L = \underbrace{min}_{x_2}\left(\left(\tilde{f}_{21}(x)\right)_\alpha^L,\left(\tilde{f}_{22}(x)\right)_\alpha^L,\dots,\left(\tilde{f}_{2m_2}(x)\right)_\alpha^L\right), \quad (18)$$

$subject\ to$

$$x \in G_0 = \left\{x \in R^2 \left| \begin{array}{l} \sum_{j=1}^n \left(\tilde{A}_{ij}\right)_\alpha^U x_j \geq \left(\tilde{b}_i\right)_\alpha^L\ (i = 1,2,\dots,r_1,r_2+1,\dots,m) \\ \sum_{j=1}^n \left(\tilde{A}_{ij}\right)_\alpha^L x_j \leq \left(\tilde{b}_i\right)_\alpha^U\ (i = r_1+1,\dots,r_2,r_2+1,\dots,m)\ x_j \geq 0, \end{array}\right.\right\} \quad (19)$$

where, the set of constraints in (19) denoted by G_0.

5. The Modified TOPSIS Approach for BL-MOP Problem with Fuzzy Parameters

In this section, a modified TOPSIS approach is presented in which a FGP approach, for more details see [5-7], is considered for solving the bi-criteria of the shortest distance from the PIS and the farthest distance from the NIS instead of the conventional max-min model. The FGP approach, developed by Mohamed [14], were recently extended to solve BL-MOP problem with fuzzy demands by Baky et al [7]. Since in decision-making situation, the aim of each DM is to achieve highest membership value (unity) of the associated fuzzy goal in order to obtain the satisfactory solution. However, in real situation, achievement of all membership values to the highest degree (unity) is not possible due to conflicting objectives. Therefore, each DM should try to maximize his or her membership function by making them as close as possible to unity by minimizing their deviational variables.

5.1. The Modified TOPSIS Model for the Upper-level MOP Problem

Consider the upper-level of minimization type problem of the α-(BL-MOP) problem (17):

$$\underbrace{min}_{x_1}\left(\left(\tilde{f}_{11}(x)\right)_\alpha^L,\left(\tilde{f}_{12}(x)\right)_\alpha^L,\dots,\left(\tilde{f}_{1m_1}(x)\right)_\alpha^L\right) \quad (20)$$

$subject\ to$

$$x \in G_0 = \left\{x \in R^2 \left| \begin{array}{l} \sum_{j=1}^n \left(\tilde{A}_{ij}\right)_\alpha^U x_j \geq \left(\tilde{b}_i\right)_\alpha^L\ (i = 1,2,\dots,r_1,r_2+1,\dots,m) \\ \sum_{j=1}^n \left(\tilde{A}_{ij}\right)_\alpha^L x_j \leq \left(\tilde{b}_i\right)_\alpha^U\ (i = r_1+1,\dots,r_2,r_2+1,\dots,m)\ x_j \geq 0, \end{array}\right.\right\} \quad (21)$$

Thus, the TOPSIS approach of Lai et al. [12] that solves single-level MODM problems is considered, in this paper, to solve the upper-level MOP problem at a desired value of α. Therefore, the bi-objective distance functions for the upper-level MOP problem follows as [1-4, 12]:

$$min\ d_P^{PIS^u}(x) \quad (22)$$

$$max\ d_P^{NIS^u}(x) \quad (23)$$

$subject\ to$

$$x \in G_0 = \left\{x \in R^2 \left| \begin{array}{l} \sum_{j=1}^n \left(\tilde{A}_{ij}\right)_\alpha^U x_j \geq \left(\tilde{b}_i\right)_\alpha^L\ (i = 1,2,\dots,r_1,r_2+1,\dots,m) \\ \sum_{j=1}^n \left(\tilde{A}_{ij}\right)_\alpha^L x_j \leq \left(\tilde{b}_i\right)_\alpha^U\ (i = r_1+1,\dots,r_2,r_2+1,\dots,m)\ x_j \geq 0, \end{array}\right.\right\} \quad (24)$$

where

$$d_P^{PIS^u}(x) = \left\{\sum_{j=1}^{m_1} w_j^p \left[\frac{\left(\tilde{f}_{1j}(x)\right)_\alpha^L - f_{1j}^*}{f_{1j}^- - f_{1j}^*}\right]^p\right\}^{\frac{1}{p}} \tag{25}$$

$$d_P^{NIS^u}(x) = \left\{\sum_{j=1}^{m_1} w_j^p \left[\frac{f_{1j}^- - \left(\tilde{f}_{11}(x)\right)_\alpha^L}{f_{1j}^- - f_{1j}^*}\right]^p\right\}^{\frac{1}{p}} \tag{26}$$

and where $f_{1j}^* = \underbrace{min}_{x\in G_0}\left(\tilde{f}_{1j}(x)\right)_\alpha^L$, is the individual positive ideal solutions, $f_{1j}^- = \underbrace{max}_{x\in G_0}\left(\tilde{f}_{1j}(x)\right)_\alpha^U$, is the individual negative ideal solutions and w_j, is the relative importance (weights) of objectives. As the problem of minimization type then $\tilde{f}_{1j}(x) = \left(\tilde{f}_{1j}(x)\right)_\alpha^L$. Let $F^{u^*} = \left(f_{11}^*, f_{12}^*, \dots, f_{1m_1}^*\right)$ and $F^{u^-} = \left(f_{11}^-, f_{12}^-, \dots, f_{1m_1}^-\right)$. Assume that the membership functions ($\mu_1(x)$ and $\mu_2(x)$) of the two objective functions are linear between $\left(d_p^u\right)^*$ and $\left(d_p^u\right)^-$ which are:

$$\left(d_P^{PIS^u}\right)^* = \underbrace{min}_{x\in G_0} d_P^{PIS^u}(x) \text{ and the solution is } x^P, \tag{27}$$

$$\left(d_P^{NIS^u}\right)^* = \underbrace{max}_{x\in G_0} d_P^{NIS^u}(x) \text{ and the solution is } x^N, \tag{28}$$

$$\left(d_P^{PIS^u}\right)^- = d_P^{PIS^u}(x^N) \text{ and } \left(d_P^{NIS^u}\right)^- = d_P^{NIS^u}(x^P) \tag{29}$$

Also, as proposed in [1-4, 12] that $\left(d_P^{PIS^u}\right)^-$ and $\left(d_P^{NIS^u}\right)^-$ can be taken as $\left(d_P^{PIS^u}\right)^- = \underbrace{max}_{x\in G_0} d_P^{PIS^u}(x)$ and $\left(d_P^{NIS^u}\right)^- = \underbrace{min}_{x\in G_0} d_P^{NIS^u}(x)$, respectively. Thus $\mu_1(x) \equiv \mu_{d_P^{PIS^u}}(x)$ and $\mu_2(x) \equiv \mu_{d_P^{NIS^u}}(x)$ can be obtained as (see: Fig. 1):

$$\mu_1(x) = \begin{cases} 1 & if\ d_P^{PIS^u}(x) < \left(d_P^{PIS^u}\right)^* \\ 1 - \dfrac{d_P^{PIS^u}(x) - \left(d_P^{PIS^u}\right)^*}{\left(d_P^{PIS^u}\right)^- - \left(d_P^{PIS^u}\right)^*} & if\ \left(d_P^{PIS^u}\right)^* \le d_P^{PIS^u}(x) \le \left(d_P^{PIS^u}\right)^- \\ 0 & if\ \left(d_P^{PIS^u}\right)^- < d_P^{PIS^u}(x) \end{cases} \tag{30}$$

$$\mu_2(x) = \begin{cases} 1 & if\ d_P^{NIS^u}(x) > \left(d_P^{NIS^u}\right)^* \\ 1 - \dfrac{\left(d_P^{NIS^u}\right)^* - d_P^{NIS^u}(x)}{\left(d_P^{NIS^u}\right)^* - \left(d_P^{NIS^u}\right)^-} & if\ \left(d_P^{NIS^u}\right)^- \le d_P^{NIS^u}(x) \le \left(d_P^{NIS^u}\right)^* \\ 0 & if\ d_P^{NIS^u}(x) < \left(d_P^{NIS^u}\right)^- \end{cases} \tag{31}$$

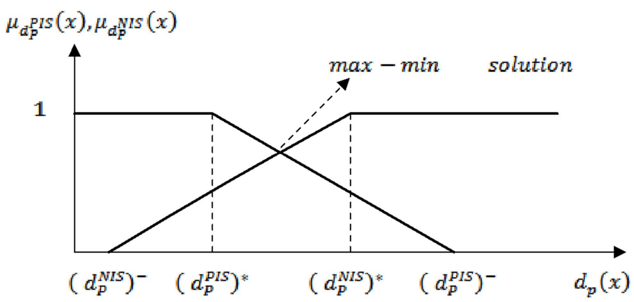

Fig. 1. *The membership functions of $\mu_{d_P^{PIS}}(x)$ and $\mu_{d_P^{NIS}}(x)$.*

Appling the FGP approach developed by Baky et al. [7], to solve the conflicting bi-objective distance functions, to achieve highest membership value (unity) of the associated fuzzy goals and obtain the satisfactory solution. So the membership functions defined in (30) and (31) for the ULDM problem having flexible membership goals with the aspired level unity can be represented as [5-7, 15]:

$$1 - \frac{d_P^{PIS^u}(x) - \left(d_P^{PIS^u}\right)^*}{\left(d_P^{PIS^u}\right)^- - \left(d_P^{PIS^u}\right)^*} + D_1^{PIS^-} - D_1^{PIS^+} = 1 \tag{32}$$

$$1 - \frac{\left(d_P^{NIS^u}\right)^* - d_P^{NIS^u}(x)}{\left(d_P^{NIS^u}\right)^* - \left(d_P^{NIS^u}\right)^-} + D_2^{NIS^-} - D_2^{NIS^+} = 1 \tag{33}$$

where $D_1^{PIS^-}, D_2^{NIS^-}$ and $D_1^{PIS^+}, D_2^{NIS^+}$ represent the under- and over-deviations from the aspired levels, respectively. The FGP approach of Mohamed [14] that solves single-level multi-objective linear programming problem is considered to solve the ULDM problem as follows:

$$min\ Z = w_1^{PIS} D_1^{PIS^+} + w_2^{NIS} D_2^{NIS^-} \tag{34}$$

subject to

$$1-\frac{d_P^{PIS^u}(x)-\left(d_P^{PIS^u}\right)^*}{\left(d_P^{PIS^u}\right)^- -\left(d_P^{PIS^u}\right)^*}+D_1^{PIS^-}-D_1^{PIS^+}=1, \qquad (35)$$

$$1-\frac{\left(d_P^{NIS^u}\right)^*-d_P^{NIS^u}(x)}{\left(d_P^{NIS^u}\right)^*-\left(d_P^{NIS^u}\right)^-}+D_2^{NIS^-}-D_2^{NIS^+}=1 \qquad (36)$$

$$\sum_{j=1}^{n}\left(\tilde{A}_{ij}\right)_\alpha^U x_j \geq \left(\tilde{b}_i\right)_\alpha^L \ (i=1,2,\dots,r_1,r_2+1,\dots,m) \qquad (37)$$

$$\sum_{j=1}^{n}\left(\tilde{A}_{ij}\right)_\alpha^L x_j \leq \left(\tilde{b}_i\right)_\alpha^U \ (i=r_1+1,\dots,r_2,r_2+1,\dots,m) \qquad (38)$$

where the numerical weights w_1^{PIS} and w_2^{NIS} represent the relative importance of achieving the aspired levels of the respective fuzzy goals subject to the constraints set in the decision situation. The weighting scheme suggested by Mohamed [14] can be used to assign the values of w_1^{PIS} and w_2^{NIS} as follows:

$$w_1^{PIS}=\frac{1}{\left(d_P^{PIS^u}\right)^- -\left(d_P^{PIS^u}\right)^*}, and\ w_2^{NIS}=\frac{1}{\left(d_P^{NIS^u}\right)^*-\left(d_P^{NIS^u}\right)^-} \qquad (39)$$

Based on the basic concepts of the bi-level programming, the ULDM sets his goals and/or decisions with possible tolerances which are described by membership functions of fuzzy set theory. According to this concept, let t_k^L and t_k^R, $k=1,2,\dots,n_1$ are the maximum acceptable negative and positive tolerance values on the decision vector considered by the ULDM, $x_1^{u^*}=\left(x_{11}^{u^*},x_{12}^{u^*},\dots,x_{1n_1}^{u^*}\right)$. The tolerances give the lower-level decision maker an extent feasible region to search for the satisfactory solution [15, 18].

The linear membership functions (Fig. 2) for each of the n_1 components of decision vector $x_1^{u^*}=\left(x_{11}^{u^*},x_{12}^{u^*},\dots,x_{1n_1}^{u^*}\right)$ controlled by the ULDM can be formulated as:

$$\mu_{x_{1k}}(x_{1k})=\begin{cases}\frac{x_{1k}-\left(x_{1k}^{u^*}-t_k^L\right)}{t_k^L}, & if\ x_{1k}^{u^*}-t_k^L\leq x_{1k}\leq x_{1k}^{u^*}\\ \frac{\left(x_{1k}^{u^*}+t_k^R\right)-x_{1k}}{t_k^R}, & if\ x_{1k}^{u^*}\leq x_{1k}\leq x_{1k}^{u^*}+t_k^R, k=1,2,\dots,n_1\\ 0, & otherwise\end{cases} \qquad (40)$$

It may be noted that, the decision maker may desire to shift the range of x_{1k}. Following Pramanik and Roy [15] and Sinha [18], this shift can be achieved.

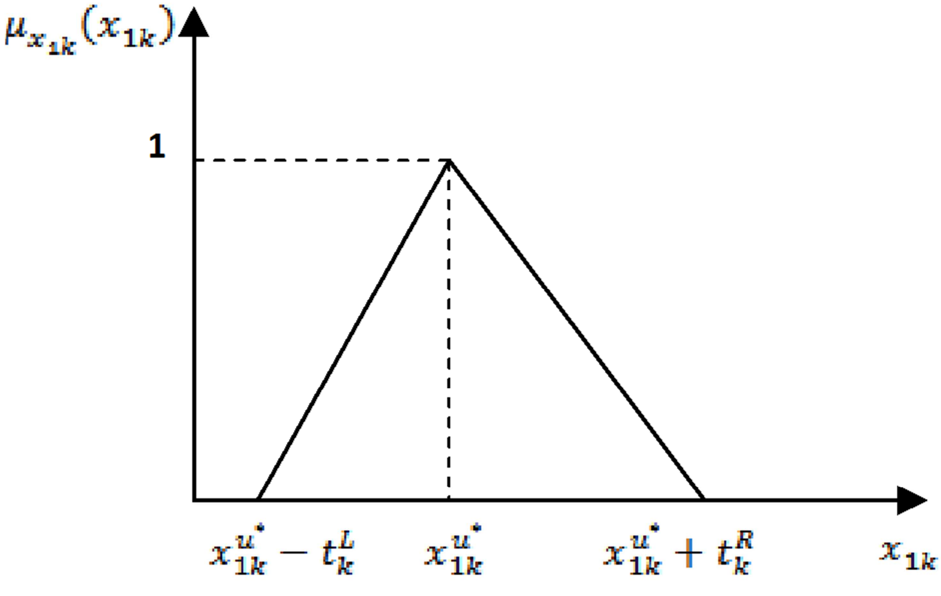

Fig. 2. *The membership function of the decision variable* x_{1k}.

5.2. The Modified TOPSIS Model for α-(BL-MOP) Problem

In order to obtain a compromise solution (satisfactory solution) to the α-(BL-MOP) problem at a specified value of α using the modified TOPSIS approach, the distance family of equation (2) to represent the distance function from the positive ideal solution, $d_P^{PIS^B}$, and the distance function from the negative ideal solution, $d_P^{NIS^B}$, are modified as both PIS (F^*) and NIS (F^-) obtained from the lower and upper bound models, respectively, of the problem to normalize the distance family. Thus, $d_P^{PIS^B}$ and $d_P^{NIS^B}$ can be proposed in this paper, for the objective functions of the upper and lower levels as follows [4, 8]:

$$d_P^{PIS^B}(x) = \left\{ \sum_{j=1}^{m_1} w_{1j}^p \left[\frac{\left(\tilde{f}_{1j}(x)\right)_\alpha^L - f_{1j}^*}{f_{1j}^- - f_{1j}^*} \right]^p + \sum_{j=1}^{m_2} w_{2j}^p \left[\frac{\left(\tilde{f}_{2j}(x)\right)_\alpha^L - f_{2j}^*}{f_{2j}^- - f_{2j}^*} \right]^p \right\}^{\frac{1}{p}} \quad (41)$$

$$d_P^{NIS^B}(x) = \left\{ \sum_{j=1}^{m_1} w_{1j}^p \left[\frac{f_{1j}^- - \left(\tilde{f}_{1j}(x)\right)_\alpha^L}{f_{1j}^- - f_{1j}^*} \right]^p + \sum_{j=1}^{m_2} w_{2j}^p \left[\frac{f_{2j}^- - \left(\tilde{f}_{2j}(x)\right)_\alpha^L}{f_{2j}^- - f_{2j}^*} \right]^p \right\}^{\frac{1}{p}} \quad (42)$$

where $w_k, k = 1,2,\dots,m_1 + m_2$ are the relative importance (weights) of objectives in both levels. $f_{ij}^* = \underbrace{min}_{x\in G_0}\left(\tilde{f}_{ij}(x)\right)_\alpha^L$, $f_{ij}^- = \underbrace{max}_{x\in G_0}\left(\tilde{f}_{ij}(x)\right)_\alpha^U, i = 1,2, j = 1,2,\dots,m_i$, and $p = 1,2,\dots,\infty$. Let $F^* = \left(f_{11}^*, f_{12}^*, \dots, f_{1m_1}^*, f_{21}^*, f_{22}^*, \dots, f_{2m_2}^*\right)$, the individual positive ideal solutions for both levels, and $F^- = \left(f_{11}^-, f_{12}^-, \dots, f_{1m_1}^-, f_{21}^-, f_{22}^-, \dots, f_{2m_2}^-\right)$, the individual negative ideal solutions for both levels. Also, for the special case of $p = \infty$, see [1-3,12] for the general form of the distance functions that can be applied to the proposed approach for solving α-(BL-MOP) problem.

In order to obtain a compromise solutions, the problem transferred into the following bi-objective problem with two commensurable (but often conflicting) objectives [1-3,12]:

$$min\ d_P^{PIS^B}(x) \quad (43)$$

$$max\ d_P^{NIS^B}(x) \quad (44)$$

$subject\ to$

$$x \in G_0 = \left\{ x \in R^2 \left| \begin{array}{l} \sum_{j=1}^{n}\left(\tilde{A}_{ij}\right)_\alpha^U x_j \ \geq \ \left(\tilde{b}_i\right)_\alpha^L \ (i = 1,2,\dots,r_1, r_2+1,\dots,m) \\ \sum_{j=1}^{n}\left(\tilde{A}_{ij}\right)_\alpha^L x_j \ \leq \ \left(\tilde{b}_i\right)_\alpha^U \ (i = r_1+1,\dots,r_2, r_2+1,\dots,m)\ x_j \geq 0, \end{array} \right. \right\} \quad (45)$$

Since these two objectives are usually conflicting to each other, it is possible to simultaneously obtain their individual optima. Thus, we can use membership functions to represent these individual optima. Assume that the membership functions ($\mu_3(x) \equiv \mu_{d_P^{PIS^B}}(x)$ and $\mu_4(x) \equiv \mu_{d_P^{NIS^B}}(x)$) of the two objective functions are linear between $\left(d_p^B\right)^*$ and $\left(d_p^B\right)^-$, they take the following form [4,8]:

$$\left(d_P^{PIS^B}\right)^* = \underbrace{min}_{x\in G_0}\ d_P^{PIS^B}(x) \text{ and the solution is } x^{PIS}, \quad (46)$$

$$\left(d_P^{NIS^B}\right)^* = \underbrace{max}_{x\in G_0}\ d_P^{NIS^B}(x) \text{ and the solution is } x^{NIS}, \quad (47)$$

$$\left(d_P^{PIS^B}\right)^- = d_P^{PIS^B}(x^{NIS}), or \left(d_P^{PIS^B}\right)^- = \underbrace{max}_{x\in G_0}\ d_P^{PIS^B}(x) \quad (48)$$

$$\left(d_P^{NIS^B}\right)^- = d_P^{NIS^B}(x^{PIS}), or \left(d_P^{NIS^B}\right)^- = \underbrace{min}_{x\in G_0}\ d_P^{NIS^B}(x) \quad (49)$$

And also, assume that $\left(d_p^B\right)^* = \left(\left(d_P^{PIS^B}\right)^*, \left(d_P^{NIS^B}\right)^*\right)$ and $\left(d_p^B\right)^- = \left(\left(d_P^{PIS^B}\right)^-, \left(d_P^{NIS^B}\right)^-\right)$. Then, based on the preference concept, we assign a larger degree to the one with shorter distance from the PIS for $\mu_3(x) \equiv \mu_{d_P^{PIS^B}}(x)$ and assign a larger degree to the one with farther distance from NIS for $\mu_4(x) \equiv \mu_{d_P^{NIS^B}}(x)$. Therefore, as shown in Fig. 1, $\mu_3(x)$ and $\mu_4(x)$ can be obtained as follows [1-3, 12]:

$$\mu_3(x) = \begin{cases} 1 & if\ d_P^{PIS^B}(x) < \left(d_P^{PIS^B}\right)^* \\ 1 - \dfrac{d_P^{PIS^B}(x) - \left(d_P^{PIS^B}\right)^*}{\left(d_P^{PIS^B}\right)^- - \left(d_P^{PIS^B}\right)^*} & if\ \left(d_P^{PIS^B}\right)^* \leq d_P^{PIS^B}(x) \leq \left(d_P^{PIS^B}\right)^- \\ 0 & if\ \left(d_P^{PIS^B}\right)^- < d_P^{PIS^B}(x) \end{cases} \quad (50)$$

$$\mu_4(x) = \begin{cases} 1 & if\ d_P^{NIS^B}(x) > \left(d_P^{NIS^B}\right)^* \\ 1 - \frac{\left(d_P^{NIS^B}\right)^* - d_P^{NIS^B}(x)}{\left(d_P^{NIS^B}\right)^* - \left(d_P^{NIS^B}\right)^-} & if\ \left(d_P^{NIS^B}\right)^- \leq d_P^{NIS^B}(x) \leq \left(d_P^{NIS^B}\right)^* \\ 0 & if\ d_P^{NIS^B}(x) < \left(d_P^{NIS^B}\right)^- \end{cases} \quad (51)$$

Finally, as discussed in section 5.1, in order to generate the satisfactory solution of, $x^* = (x_1^*, x_2^*)$, the final proposed model of the modified TOPSIS that includes the membership function equation (40) for the upper-level decision variables vector, $x_1^{u^*} = \left(x_{11}^{u^*}, x_{12}^{u^*}, \ldots, x_{1n_1}^{u^*}\right)$, follows as:

$$min\ Z = w_3^{PIS}\, D_3^{PIS^+} + w_4^{NIS} D_4^{NIS^-} + \sum_{k=1}^{n_1}[w_{1k}^L(D_{1k}^{L-} + D_{1k}^{L+}) + w_{1k}^R(D_{1k}^{R-} + D_{1k}^{R+})] \quad (52)$$

$$subject\ to$$

$$1 - \frac{d_P^{PIS^B}(x) - \left(d_P^{PIS^B}\right)^*}{\left(d_P^{PIS^B}\right)^- - \left(d_P^{PIS^B}\right)^*} + D_3^{PIS^-} - D_3^{PIS^+} = 1 \quad (53)$$

$$1 - \frac{\left(d_P^{NIS^B}\right)^* - d_P^{NIS^B}(x)}{\left(d_P^{NIS^B}\right)^* - \left(d_P^{NIS^B}\right)^-} + D_4^{NIS^-} - D_4^{NIS^+} = 1 \quad (54)$$

$$\frac{x_{1k} - \left(x_{1k}^{u^*} - t_k^L\right)}{t_k^L} + D_{1k}^{L-} - D_{1k}^{L+} = 1, k = 1,2 \ldots, n_1. \quad (55)$$

$$\frac{\left(x_{1k}^{u^*} + t_k^R\right) - x_{1k}}{t_k^R} + D_{1k}^{R-} - D_{1k}^{R+} = 1, k = 1,2 \ldots, n_1. \quad (56)$$

$$\sum_{j=1}^{n}\left(\tilde{A}_{ij}\right)_\alpha^U x_j \geq \left(\tilde{b}_i\right)_\alpha^L (i = 1,2, \ldots, r_1, r_2 + 1, \ldots, m) \quad (57)$$

$$\sum_{j=1}^{n}\left(\tilde{A}_{ij}\right)_\alpha^L x_j \leq \left(\tilde{b}_i\right)_\alpha^U (i = r_1 + 1, \ldots, r_2, r_2 + 1, \ldots, m)\ x_j \geq 0, (j = 1,2) \quad (58)$$

where $D_{1k}^- = (D_{1k}^{L-}, D_{1k}^{R-}), D_{1k}^+ = (D_{1k}^{L+}, D_{1k}^{R+})$ and $D_3^{PIS^-}, D_4^{NIS^-}, D_{1k}^{L-}, D_{1k}^{R-}, D_3^{PIS^+}, D_4^{NIS^+}, D_{1k}^{L+}, D_{1k}^{R+} \geq 0$ with $D_3^{PIS^-} \times D_3^{PIS^+} = 0, D_4^{NIS^-} \times D_4^{NIS^+} = 0, D_{1k}^{L-} \times D_{1k}^{L+} = 0$ and $D_{1k}^{R-} \times D_{1k}^{R+} = 0, k = 1,2, \ldots, n_1$, represent the under- and over-deviation, respectively, from the aspired levels. Also Z represents the fuzzy achievement function. Again, to assess the relative importance of the fuzzy goals properly, the weighting scheme suggested by Mohamed [14] can be used to assign the values to $w_3^{PIS}, w_4^{NIS}, w_{1k}^L$ and w_{1k}^R as:

$$w_3^{PIS} = \frac{1}{\left(d_P^{PIS^B}\right)^- - \left(d_P^{PIS^B}\right)^*}, and\ w_4^{NIS} = \frac{1}{\left(d_P^{NIS^B}\right)^* - \left(d_P^{NIS^B}\right)^-} \quad (59)$$

$$w_{1k}^L = \frac{1}{t_k^L}, and\ w_{1k}^R = \frac{1}{t_k^R}, k = 1,2 \ldots, n_1. \quad (60)$$

5.3. The Modified TOPSIS Algorithm for BL-MOP Problem with Fuzzy Parameters

The modified TOPSIS model (52)-(58) provides a satisfactory decision for the two DMs at the two levels. Following the above discussion, the algorithm for the proposed modified TOPSIS approach for solving BL-MOP problem with fuzzy parameters is given as follows:

Step 1. Formulate the deterministic model of the BL-MOP problem, for a prescribed value of α.

Step 2. Determine the individual maximum and minimum values from the upper and lower bound of α-level.

Step 3. Construct the PIS payoff table of the ULDM problem (20)&(21) from the lower bound model and obtain $F^{u^*} = \left(f_{11}^*, f_{12}^*, \ldots, f_{1m_1}^*\right)$, the individual positive ideal solutions.

Step 4. Construct the NIS payoff table of the ULDM problem (20)&(21) from the upper bound model and obtain $F^{u^-} = \left(f_{11}^-, f_{12}^-, \ldots, f_{1m_1}^-\right)$, the individual negative ideal solutions.

Step 5. Use Eq. (25)&(26) to construct $d_P^{PIS^u}(x)$ and $d_P^{NIS^u}(x)$.

Step 6. Ask the DM to select p, $\{p = 1,2, \ldots, \infty\}$.

Step 7. Construct the payoff table of problem (27)-(29) and obtain $\left(d_p^u\right)^*$ and $\left(d_p^u\right)^-$.

Step 8. Elicit the membership functions $\mu_{d_P^{PIS^u}}(x)$ and $\mu_{d_P^{NIS^u}}(x)$.

Step 9. Formulate and solve the model (34)-(38) for the ULDM problem to get $x^{u^*} = (x_1^{u^*}, x_2^{u^*})$, $x_1^{u^*} = (x_{11}^{u^*}, x_{12}^{u^*}, \dots, x_{1n_1}^{u^*})$.

Step 10. Set the maximum negative and positive tolerance values on the decision vector $x_1^{u^*} = (x_{11}^{u^*}, x_{12}^{u^*}, \dots, x_{1n_1}^{u^*})$, t_k^L and $t_k^R, k = 1,2, \dots, n_1$.

Step 11. Construct the PIS payoff table of the α-(BL-MOP) problem from the lower bound model and obtain $F^* = (f_{11}^*, f_{12}^*, \dots, f_{1m_1}^* f_{21}^*, f_{22}^*, \dots, f_{2m_2}^*)$, the individual positive ideal solutions for both levels.

Step 12. Construct the NIS payoff table of the α-(BL-MOP) problem from the upper bound model and obtain $F^- = (f_{11}^-, f_{12}^-, \dots, f_{1m_1}^- f_{21}^-, f_{22}^-, \dots, f_{2m_2}^-)$, the individual negative ideal solutions for both levels.

Step 13. Use Eqs. (41) and (42) to construct $d_P^{PIS^B}(x)$ and $d_P^{NIS^B}(x)$, respectively.

Step 14. Construct the payoff table of problem (43)-(45) and obtain $(d_p^B)^*$ and $(d_p^B)^-$.

Step 15. Elicit the membership functions $\mu_{d_P^{PIS^B}}(x)$ and $\mu_{d_P^{NIS^B}}(x)$.

Step 16. Elicit the membership functions $\mu_{x_{1k}}(x_{1k}), k = 1,2, \dots, n_1$.

Step 17. Formulate and solve the model (52)-(58) for the α-(BL-MOP) problem to get $x^* = (x_1^*, x_2^*)$.

Step 18. If the DM is satisfied with the candidate solution in Step 17, go to Step 20, otherwise go to Step 19.

Step 19. Modify the maximum negative and positive tolerance values on the decision vector $x_1^{u^*} = (x_{11}^{u^*}, x_{12}^{u^*}, \dots, x_{1n_1}^{u^*})$, t_k^L and $t_k^R, k = 1,2, \dots, n_1$, go to Step 16.

Step 20. Stop with a satisfactory solution, $x^* = (x_1^*, x_2^*)$, to the α-(BL-MOP) problem.

6. Illustrative Numerical Example

The following numerical example studied in [7, 8, 16] is considered to illustrate the modified TOPSIS approach for solving BL-MOP problem with fuzzy parameters.

[1st level]

$$\underbrace{min}_{x_1,x_2} \begin{pmatrix} \tilde{f}_{11}(x)=(x_1+\tilde{3}x_2+\tilde{2}x_3+\tilde{3}x_4), \\ \tilde{f}_{12}(x)=(\tilde{2}x_1+\tilde{9}x_2+\tilde{3}x_3+\tilde{5}x_4), \\ \tilde{f}_{13}(x)=(\tilde{3}x_1+\tilde{9}x_2+\tilde{9}x_3+ x_4) \end{pmatrix}$$

[2nd level]

$$\underbrace{min}_{x_3,x_4} \begin{pmatrix} \tilde{f}_{21}(x)=(\tilde{6}x_1+\tilde{3}x_2+\tilde{2}x_3+\tilde{2}x_4), \\ \tilde{f}_{22}(x)=(\tilde{5}x_1+\tilde{9}x_2-\tilde{9}x_3+\tilde{6}x_4) \end{pmatrix}$$

subject to

$$\tilde{3}x_1 - x_2 + x_3 + \tilde{3}x_4 \le \widetilde{48}, \tilde{2}x_1 + \tilde{4}x_2 + \tilde{2}x_3 - \tilde{2}x_4 \le \widetilde{35},$$

$$x_1 + \tilde{2}x_2 - x_3 + x_4 \ge \widetilde{30}, x_1, x_2, x_3, x_4 \ge 0.$$

Here, the fuzzy numbers are assumed to be triangular fuzzy numbers and are given as follows:

$$\tilde{2} = (0,2,3), \tilde{3} = (2,3,4), \tilde{4} = (3,4,5), \tilde{5} = (4,5,6), \tilde{6} = (5,6,7), \tilde{8} = (6,8,10), \tilde{9} = (8,9,10), \widetilde{30} = (28,30,32). \widetilde{35} = (33,35,37), \widetilde{48} = (45,48,49).$$

Since the problem is minimization-type then replacing the fuzzy coefficient by their lower bound α-cuts, for α=0.5, the BL-MOP problem with fuzzy parameters reduces to a deterministic BL-MOP problem as follows:

[1st level]

$$\underbrace{min}_{x_1,x_2} \begin{pmatrix} \left(\tilde{f}_{11}(x)\right)_{0.5}^{L} = x_1+ 2.5x_2+ x_3+ 2.5x_4 \\ \left(\tilde{f}_{12}(x)\right)_{0.5}^{L} = x_1+8.5x_2+2.5x_3+4.5x_4 \\ \left(\tilde{f}_{13}(x)\right)_{0.5}^{L} = 2.5x_1+ 8.5x_2+ 8.5x_3+x_4 \end{pmatrix}$$

[2nd level]

$$\underbrace{min}_{x_3,x_4} \begin{pmatrix} \left(\tilde{f}_{21}(x)\right)_{0.5}^{L} =5.5x_1+ 2.5x_2+ x_3+ x_4 \\ \left(\tilde{f}_{22}(x)\right)_{0.5}^{L} =4.5x_1+8.5x_2-8.5x_3+5.5x_4 \end{pmatrix}$$

subject to

$$x \in G_0 = \begin{Bmatrix} 2.5x_1 - x_2 + x_3 + 2.5x_4 \le 48.5 \\ x_1 + 2.5x_2 - x_3 + x_4 \ge 29 \\ x_1 + 3.5x_2 + x_3 - x_4 \le 36 \\ x_1, x_2, x_3, x_4 \ge 0. \end{Bmatrix}$$

The individual minimum and maximum of each of the objective functions at both levels calculated from the lower and upper bound α-cut model are given in Table 1.

Table 1. *Minimum and maximum individual optimal solutions.*

Objective Function	$\left(\tilde{f}_{11}(x)\right)_{0.5}^{L,U}$	$\left(\tilde{f}_{12}(x)\right)_{0.5}^{L,U}$	$\left(\tilde{f}_{13}(x)\right)_{0.5}^{L,U}$	$\left(\tilde{f}_{21}(x)\right)_{0.5}^{L,U}$	$\left(\tilde{f}_{22}(x)\right)_{0.5}^{L,U}$
$min\left(\tilde{f}_{ij}(x)\right)_{0.5}^{L}$	29	48.862	48.862	29	55.875
$max\left(\tilde{f}_{ij}(x)\right)_{0.5}^{U}$	155.47	315.79	268.46	152.1	342.34

We first obtain PIS and NIS payoff tables for the ULMD problem from the lower and upper bound model respectively (Table 2 and 3):

Table 2. *PIS payoff table of the ULDM problem.*

Objective Function	$\left(\tilde{f}_{11}(x)\right)_{0.5}^{L}$	$\left(\tilde{f}_{12}(x)\right)_{0.5}^{L}$	$\left(\tilde{f}_{13}(x)\right)_{0.5}^{L}$	x_1	x_2	x_3	x_4
$min\left(\tilde{f}_{11}(x)\right)_{0.5}^{L}$	29*	71	88.25	11.5	7	0	0
$min\left(\tilde{f}_{12}(x)\right)_{0.5}^{L}$	29.005	48.862*	79.96	20.73	3.31	0	0
$min\left(\tilde{f}_{13}(x)\right)_{0.5}^{L}$	60.1	121.4	48.862*	0	3.31	0	20.73

$$F^{u^*} = (f_{11}^*, f_{12}^*, f_{13}^*) = (29, 48.862, 48.862)$$

Table 3. *NIS payoff table of the ULDM problem.*

Objective Function	$\left(\tilde{f}_{11}(x)\right)_{0.5}^{U}$	$\left(\tilde{f}_{12}(x)\right)_{0.5}^{U}$	$\left(\tilde{f}_{13}(x)\right)_{0.5}^{U}$	x_1	x_2	x_3	x_4
$max\left(\tilde{f}_{11}(x)\right)_{0.5}^{U}$	155.47*	315.79	196.32	0	17.87	0	26.55
$max\left(\tilde{f}_{12}(x)\right)_{0.5}^{U}$	155.47	315.79*	196.32	0	17.87	0	26.55
$max\left(\tilde{f}_{13}(x)\right)_{0.5}^{U}$	138.11	253.67	268.46*	0	10.38	15.85	17.5

$$F^{u^-} = (f_{11}^-, f_{12}^-, f_{13}^-) = (155.47, 315.79, 268.46)$$

Assume that $w_i = \frac{1}{3}, i = 1, 2, 3$, then the equations for $d_P^{PIS^u}(x)$ and $d_P^{NIS^u}(x)$ when $p = 2$ are:

$$F_1^u = d_2^{PIS^u}(x) = \begin{Bmatrix} 0.0000069[(x_1 + 2.5x_2 + x_3 + 2.5x_4) - 29]^2 + \\ 0.00000156[(x_1 + 8.5x_2 + 2.5x_3 + 4.5x_4) - 48.862]^2 + \\ 0.0000023[(2.5x_1 + 8.5x_2 + 8.5x_3 + x_4) - 48.862]^2 \end{Bmatrix}^{1/2}$$

$$F_2^u = d_2^{NIS^u}(x) = \begin{Bmatrix} 0.0000069[155.47 - (x_1 + 2.5x_2 + x_3 + 2.5x_4)]^2 + \\ 0.00000156[315.79 - (x_1 + 8.5x_2 + 2.5x_3 + 4.5x_4)]^2 + \\ 0.0000023[268.46 - (2.5x_1 + 8.5x_2 + 8.5x_3 + x_4)]^2 \end{Bmatrix}^{1/2}$$

Next to formulate model (34)-(38) we determine the following $max\ F_1^u = 0.41, min\ F_1^u = 0.044$, and $max\ F_2^u = 0.553, min\ F_2^u = 0.188$. Thus we have $d_2^{u^*} = (0.044, 0.553)$ and $d_2^{u^-} = (0.41, 0.188)$, therefore, the membership functions $\mu_1(x)\ and\ \mu_2(x)$ can be obtained as:

$$\mu_{F_1^u}(x) = 1.12 - 2.73F_1^u$$

$$\mu_{F_2^u}(x) = -0.515 + 2.74F_2^u$$

Then, the modified TOPSIS formulation for the ULDM problem is obtained as follows:

$$min\ Z = 2.73\, D_1^{PIS^+} + 2.74 D_2^{NIS^-}$$

$subject\ to$

$$1.12 - 2.73F_1^u + D_1^{PIS^-} - D_1^{PIS^+} = 1$$

$$-0.515 + 2.74F_2^u + D_2^{NIS^-} - D_2^{NIS^+} = 1$$

$$2.5x_1 - x_2 + x_3 + 2.5x_4 \le 48.5$$

$$x_1 + 2.5x_2 - x_3 + x_4 \ge 29$$

$$x_1 + 3.5x_2 + x_3 - x_4 \le 36$$

$$x_1, x_2, x_3, x_4 \ge 0.\ D_1^{PIS^-}, D_1^{PIS^+}, D_2^{NIS^-}, D_2^{NIS^+} \ge 0,$$

The optimal solution of the ULDM problem is achieved at $x^{u^*} = (20.678, 3.315, 0.0096, 0.044)$. Let the upper-level DM decide $x_1^{u^*} = 20.678$, and $x_2^{u^*} = 3.315$, with positive tolerances $t_1^R = t_2^R = 0.5$ and weights of $w_{1k}^R = \frac{1}{0.5} = 2$, (one sided membership function [23, 28]).

We first obtain PIS and NIS payoff tables for the second level MOP problem from the lower and upper bound model respectively (Tables 4 and 5):

Table 4. *PIS payoff table of the LLDM problem.*

Objective Function	$\left(\tilde{f}_{21}(x)\right)_{0.5}^{L}$	$\left(\tilde{f}_{22}(x)\right)_{0.5}^{L}$	x_1	x_2	x_3	x_4
$min\left(\tilde{f}_{21}(x)\right)_{0.5}^{L}$	29*	102.6	0	10.83	0	1.917
$min\left(\tilde{f}_{22}(x)\right)_{0.5}^{L}$	60.16	55.875*	0	10.83	15.58	17.5

$$F^{l^*} = (f_{21}^*, f_{22}^*) = (29, 55.875)$$

Table 5. *NIS payoff table of the LLDM problem.*

Objective Function	$\left(\tilde{f}_{21}(x)\right)_{0.5}^{U}$	$\left(\tilde{f}_{22}(x)\right)_{0.5}^{U}$	x_1	x_2	x_3	x_4
$max\left(\tilde{f}_{21}(x)\right)_{0.5}^{U}$	152.1*	156.52	21.1	4.26	0	0
$max\left(\tilde{f}_{22}(x)\right)_{0.5}^{U}$	128.92	342.34*	0	17.87	0	26.55

$$F^{l^-} = (f_{21}^-, f_{22}^-) = (152.1, 342.34)$$

Assume that $w_i = \frac{1}{5}$ $i = 1,2,3,4,5$, then the equations for $d_P^{PIS^B}(x)$ and $d_P^{NIS^B}(x)$ when $p = 2$ are:

$$P_1^B = d_2^{PIS^B}(x) = \begin{Bmatrix} 0.0000025[(x_1 + 2.5x_2 + x_3 + 2.5x_4) - 29]^2 + \\ 0.00000056\ [(x_1 + 8.5x_2 + 2.5x_3 + 4.5x_4) - 48.862]^2 + \\ 0.00000083[(2.5x_1 + 8.5x_2 + 8.5x_3 + x_4) - 48.862]^2 + \\ 0.0000026[(5.5x_1 + 2.5x_2 + x_3 + x_4) - 29]^2 + \\ 0.00000049[(4.5x_1 + 8.5x_2 - 8.5x_3 + 5.5x_4) - 55.875]^2 \end{Bmatrix}^{1/2}$$

$$P_2^B = d_2^{NIS^B}(x) = \begin{Bmatrix} 0.0000025[155.47 - (x_1 + 2.5x_2 + x_3 + 2.5x_4)]^2 + \\ 0.00000056\ [315.79 - (x_1 + 8.5x_2 + 2.5x_3 + 4.5x_4)]^2 + \\ 0.00000083[268.46 - (2.5x_1 + 8.5x_2 + 8.5x_3 + x_4)]^2 + \\ 0.0000026[152.1 - (5.5x_1 + 2.5x_2 + x_3 + x_4)]^2 + \\ 0.00000049[342.34 - (4.5x_1 + 8.5x_2 - 8.5x_3 + 5.5x_4)]^2 \end{Bmatrix}^{1/2}$$

Next we determine the following $max\ P_1^B = 0.303, min\ P_1^B = 0.065$, and $max\ P_2^B = 0.396, min\ P_2^B = 0.175$. Thus we have $d_2^{B^*} = (0.065, 0.396)$ and $d_2^{B^-} = (0.303, 0.175)$, therefore, the membership functions $\mu_3(x)$ *and* $\mu_4(x)$ can be obtained as:

$$\mu_{P_1^B}(x) = 1.27 - 4.23P_1^B$$

$$\mu_{P_2^B}(x) = -0.792 + 4.52P_2^B$$

Finally, the modified TOPSIS formulation for the α-(BL-MOP) problem is obtained as:

$$min\ Z = 4.23D_3^{PIS^+} + 4.52D_4^{NIS^-} + 2D_1^{R-} + 2D_1^{R+} + 2D_2^{R-} + 2D_2^{R+}$$

subject to

$$1.27 - 4.23P_1^B + D_3^{PIS^-} - D_3^{PIS^+} = 1$$

$$-0.792 + 4.52P_2^B + D_4^{NIS^-} - D_4^{NIS^+} = 1$$

$$42.356 - 2x_1 + D_1^{R-} - D_1^{R+} = 1$$

$$7.64 - 2x_2 + D_2^{R-} - D_2^{R+} = 1$$

$$2.5x_1 - x_2 + x_3 + 2.5x_4 \le 48.5$$

$$x_1 + 2.5x_2 - x_3 + x_4 \ge 29$$

$$x_1 + 3.5x_2 + x_3 - x_4 \le 36$$

$$x_1, x_2, x_3, x_4 \ge 0.\ D_1^{PIS^-}, D_1^{PIS^+}, D_2^{NIS^-}, D_2^{NIS^+} \ge 0,$$

$$D_3^{PIS^-} \times D_3^{PIS^+} = 0,\ D_4^{NIS^-} \times D_4^{NIS^+} = 0, \text{ and } D_k^{R-} \times D_k^{R+} = 0\ k = 1,2$$

The satisfactory solution of the α-(BL-MOP) problem is $x^* = (20.68, 3.32, 0, 0.02)$ with objective function values $f_{11} = 29.03, f_{12} = 48.99, f_{13} = 79.94, f_{21} = 122.1$, and $f_{22} = 121.4$, and their corresponding membership functions are $\mu_{11} = 0.9997, \mu_{12} = 0.9995, \mu_{13} = 0.858, \mu_{21} = 0.244$, and $\mu_{22} = 0.771$, respectively.

The comparison given in Table 6 between the modified TOPSIS approach and the TOPSIS approach [8] shows that the modified TOPSIS approach is greatly preferred than the later approach.

Table 6. *Comparison between the modified TOPSIS approach and the TOPSIS approach [8].*

The modified TOPSIS approach		The TOPSIS approach [8]		The optimal solution
$f_{11} = 29.03$	$\mu_{11} = 0.999$	$f_{11} = 57.19$	$\mu_{11} = 0.78$	$f_{11} = 29$
$f_{12} = 48.99$	$\mu_{12} = 0.999$	$f_{12} = 114.93$	$\mu_{12} = 0.75$	$f_{12} = 48.862$
$f_{13} = 79.94$	$\mu_{13} = 0.858$	$f_{13} = 51.88$	$\mu_{13} = 0.986$	$f_{13} = 48.862$
$f_{21} = 122.1$	$\mu_{21} = 0.244$	$f_{21} = 37.16$	$\mu_{21} = 0.94$	$f_{21} = 29$
$f_{22} = 121.4$	$\mu_{22} = 0.771$	$f_{22} = 140.1$	$\mu_{22} = 0.71$	$f_{22} = 55.875$

For indicating the merits of the modified TOPSIS approach, the comparison is given in Table 7 among modified TOPSIS approach, the method of Pramanik [16] and the FGP algorithm [7] by Baky et al. to solve the BL-MOP problem with fuzzy parameters shows that the values of objective and membership functions of the BL-MOP problem with fuzzy parameters obtained from the modified TOPSIS are more preferred than that given by the TOPSIS approach and FGP algorithm. Since the modified TOPSIS approach combines the advantages of both TOPSIS and FGP approach. And because the TOPSIS approach transfers q objectives which are conflicting and non-commensurable into two objectives (the shortest distance from the PIS and the longest distance from the NIS), those which are commensurable and most of time conflicting. Hence, FGP approach solves the bi-objective problem to obtain the satisfactory solution.

Table 7. *Comparison between the modified TOPSIS approach, the method of Pramanik et al. and FGP algorithm.*

The modified TOPSIS approach		The method of Pramanik [16]		The FGP algorithm [7]	
$f_{11} = 29.03$	$\mu_{11} = 0.999$	$f_{11} = 37$	$\mu_{11} = 0.902$	$f_{11} = 29$	$\mu_{11} = 1$
$f_{12} = 48.99$	$\mu_{12} = 0.999$	$f_{12} = 90$	$\mu_{12} = 0.815$	$f_{12} = 70.95$	$\mu_{12} = 0.917$
$f_{13} = 79.94$	$\mu_{13} = 0.858$	$f_{13} = 108.25$	$\mu_{13} = 0.692$	$f_{13} = 88.27$	$\mu_{13} = 0.821$
$f_{21} = 122.1$	$\mu_{21} = 0.244$	$f_{21} = 78.25$	$\mu_{21} = 0.496$	$f_{21} = 80.9$	$\mu_{21} = 0.578$
$f_{22} = 121.4$	$\mu_{22} = 0.771$	$f_{22} = 105.5$	$\mu_{22} = 0.795$	$f_{22} = 111.27$	$\mu_{22} = 0.81$

7. Conclusion and Summary

This paper reveals how the concept of modified TOPSIS approach can be efficiently used to solve the BL-MOP problem with fuzzy parameters. In order to obtain a compromise (satisfactory) solution to the BL-MOP problem with fuzzy parameters using the modified TOPSIS approach, firstly the crisp model is obtained at an α-level. Then, the distance function from the positive ideal solution and the distance function from the negative ideal solution are developed. Then, the bi-objective problem can be formulated by using the membership functions of the fuzzy set theory to represent the satisfaction level for both criteria. Hence, in the modified TOPSIS approach, the FGP approach is used to solve the confliction between the new criteria instead of the conventional max-min decision model. An illustrative numerical example is given to demonstrate the efficiency of the proposed modified TOPSIS approach for BL-MOP problem with fuzzy parameters. The comparison among the proposed modified TOPSIS approach and the existing methods is given in Tables 6&7. Finally, it is hoped that the concept of solving BL-MOP problem with fuzzy parameters presented here can contribute to future studies in the other fields of MODM problems.

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