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I-Statistically Pre-Cauchy Triple Sequences of Fuzzy Real Numbers

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Abstract: In this article, the notion of *I*-statistically pre-Cauchy sequence of fuzzy real numbers having multiplicity greater than two is introduced. We establish the criterion for any arbitrary triple sequence of fuzzy numbers to be *I*-statistically pre-Cauchy. It is shown that an *I*-statistically convergent sequence of fuzzy numbers is *I*-statistically pre-Cauchy. Moreover a necessary and sufficient condition for a bounded triple sequence of fuzzy real numbers to be I-pre-Cauchy is established.

Keywords: Ideal, Filter, Statistical Convergence, Ideal Convergence, I-Statistical Convergence, Triple Sequence of Fuzzy Numbers, *I*-statistical Pre-Cauchy, Orlicz Function

1. Introduction

Fuzzy set theory, compared to other mathematical theories, is perhaps the most easily adaptable theory to practice. The main reason is that a fuzzy set has the property of relativity, variability, and inexactness in the definition of its elements. Instead of defining an entity in calculus by assuming that its role is exactly known, we can use fuzzy sets to define the same entity by allowing possible deviations and inexactness in its role. Fuzzy set theory has become an area of active area of research in science and engineering for the last 46 years. While studying fuzzy topological spaces, many situations are faced, where we need to deal with convergence of fuzzy numbers. The concept of fuzzy set is first introduced by Zadeh [39] in 1965. Later on, the sequence of fuzzy numbers is discussed by several mathematicians such as Matloka [20], Nanda [22], Savas [30] and many others.

The notion of statistical convergence was first introduced by Fast [13]. After then it was studied by many researchers like Šalát [27], Fridy [14], Connor [4], Maddox [19], Kwon [18], Savas [29]. Different classes of statistically convergent sequences were introduced and investigated by Tripathy and Sen [37], Tripathy and Sarma [36] etc. Móricz [21] extended statistical convergence from single to multiple real sequences. Nuray and Savas [24] first defined the concepts of statistical convergence and statistically Cauchy for sequences of fuzzy numbers. The notion of statistically pre-Cauchy for real sequences was introduced by Connor, Fridy and Kline [5]. More works on statistically pre-Cauchy sequences are found in Khan and Lohani [16], Dutta [10], Dutta and Tripathy [9], Das *et al*. [7] etc.

Agnew [1] studied the summability theory of multiple sequences and obtained certain theorems which have already been proved for double sequences by the author himself. The different types of notions of triple sequences was introduced and investigated at the initial stage by Sahiner *et al*. [25], Sahiner and Tripathy [26]. Recently Savas and Esi [32] have introduced statistical convergence of triple sequences on probabilistic normed space. Later on, Esi [11] have introduced statistical convergence of triple sequences in topological groups. Some more works on triple sequences are found on Kumar *et al.* [17], Esi [12], Dutta *et al.* [8], Tripathy and Dutta [34], Tripathy and Goswami [35], Nath and Roy [23] etc.

The idea of statistical convergence is extended to *I*-convergence in case of real's by using the notion of ideals of *N*. Kostyrko, Šalát and Wilczyński [15] introduced the concept of ideal convergence for single sequences in 2000-2001. Later on it was further developed by Šalát *et al*. [28], Das *et al*. [6], Sen and Roy [33] and many others. Savas and Das [31] used ideals to introduce the concept of *I*-statistical convergence for real's which have extended the notion of statistical convergence. The notion of *I*-statistically pre-

Cauchy sequences are also introduced by them. In the present article, we have extended these results to introduce the concept of *I*-statistically pre-Cauchy triple sequence of fuzzy real numbers.

A fuzzy real number on R is a mapping $X: R \to L(=[0,1])$ associating each real number $t \in R$ with its grade of membership $X(t)$. Every real number r can be expressed as a fuzzy real number $\bar{r}$ as follows:

$$\bar{r}(t) = \begin{cases} 1 & if \quad t = r \\ 0 & otherwise \end{cases}$$

The α-level set of a fuzzy real number X, $0 < \alpha \le 1$, denoted by $[X]^\alpha$ is defined as

$$[X]^\alpha = \{t \in R : X(t) \ge \alpha\}.$$

A fuzzy real number X is called convex if $X(t) \ge X(s) \wedge X(r) = \min\ (X(s), X(r))$, where $s < t < r$. If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called normal. A fuzzy real number X is said to be upper semi-continuous if for each $\varepsilon > 0$, $X^{-1}[0, a+\varepsilon))$, for all $a \in L$ is open in the usual topology of R. The set of all upper semi continuous, normal, convex fuzzy number is denoted by $R(L)$. The additive identity and multiplicative identity in $R(L)$ are denoted by $\bar{0}$ and $\bar{1}$ respectively.

Let D be the set of all closed bounded intervals $X = \left[X^L, X^R\right]$ on the real line R. Then

$X \le Y$ if and only if $X^L \le Y^L$ and $X^R \le Y^R$. Also let $d(X,Y) = \max\left(|X^L - X^R|, |Y^L - Y^R|\right)$.

Then (D, d) is a complete metric space.

Let $\bar{d}: R(L) \times R(L) \to R$ be defined by $\bar{d}(X,Y) = \sup_{0 \le \alpha \le 1} d([X]^\alpha, [Y]^\alpha)$, for $X, Y \in R(L)$.

Then $\bar{d}$ defines a metric on $R(L)$.

Recent Development on Fuzzy Distance Measure are found in Beigi *et al.* ([2], [3]).

2. Preliminaries and Background

In this section, some notations and basic definitions which will be used in this article are recalled.

A triple sequence can be defined as a function $x: N \times N \times N \to R(C)$, where N, R and C denote the sets of natural numbers, real numbers and complex numbers respectively.

The notion of statistical convergence for triple sequences depends on the density of the subsets of $N \times N \times N$. A subset E of $N \times N \times N$ is said to have density or asymptotic density $\delta_3(E)$, if

$$\delta_3(E) = \lim_{p,q,r \to \infty} \frac{1}{pqr} \sum_{i \le p} \sum_{j \le q} \sum_{k \le r} \chi_E(i,j,k), \text{ exists, where } \chi_E$$

is the characteristic function of E.

The notion of Ideal convergence depends on the structure of the ideal *I* of the subset of the set of natural numbers *N*.

Let *X* be a non empty set. A non-void class $I \subseteq 2^X$ (power set of *X*) is said to be an ideal if *I* is additive and hereditary, *i.e.* if *I* satisfies the following conditions:

(i). $A, B \in I \Rightarrow A \cup B \in I$ and

(ii). $A \in I$ *and* $B \subseteq A \Rightarrow B \in I$.

A non-empty family of sets $F \subseteq 2^X$ is said to be a filter on X if

(i). $\varnothing \notin$ F

(ii). A, B $\in$ F $\Rightarrow$ A $\cap$ B $\in$ F

(iii). A $\in$ F and A $\subseteq$ B $\Rightarrow$ B $\in$ F.

For any ideal *I*, there is a filter *F*(*I*) given by $F(I) = \{K \subseteq N : N \backslash K \in I\}$.

An ideal $I \subseteq 2^X$ is said to be non-trivial if I $\neq \varnothing$ and X $\notin$ I.

The details about the ideals of $2^{N \times N}$ are introduced and investigated by Tripathy and Tripathy [38].

Throughout the article, the ideals of $2^{N \times N \times N}$ will be denoted by I_3.

Example 2.1. Let $I_3(\rho) \subset 2^{N \times N \times N}$ i.e. the class of all subsets of $N \times N \times N$ of zero natural density.

Then $I_3(\rho)$ is an ideal of $2^{N \times N \times N}$.

Example 2.2. Let $I_3(P)$ be the class of all subsets of $N \times N \times N$ such that $D \in I_3(P)$ implies that there exists n_0, $l_0, k_0 \in N$ such that

$$D \subseteq N \times N \times N - \{(n,l,k) \in N \times N \times N : n \ge n_0,\ l \ge l_0,\ k \ge k_0\}.$$

Then $I_3(P)$ is an ideal of $2^{N \times N \times N}$.

A fuzzy real-valued triple sequence $X = \langle X_{ijk} \rangle$ is a triple infinite array of fuzzy real numbers X_{ijk} for all $i, j, k \in N$ and is denoted by $\langle X_{ijk} \rangle$ where $X_{ijk} \in R(L)$.

A fuzzy real-valued triple sequence $X = \langle X_{ijk} \rangle$ is said to be statistically convergent to the fuzzy real number L, if for all $\varepsilon > 0$,

$$\lim_{m,n,l \to \infty} \frac{1}{mnl} \left| \{(i,j,k) : \bar{d}(X_{ijk}, L) \ge \varepsilon, i \le m; j \le n; k \le l\} \right| = 0,$$

where $|.|$ denote the cardinality of the enclosed set and we write $stat_3 - \lim X_{ijk} = L$.

A fuzzy real-valued triple sequence $X = \langle X_{ijk} \rangle$ is said to be I-convergent to the fuzzy real number L, if for each $\varepsilon > 0$, the set $\left\{(i,j,k) \in N \times N \times N : \bar{d}(X_{ijk}, L) \ge \varepsilon\right\} \in I_3$.

A fuzzy real-valued triple sequence $X = \langle X_{ijk} \rangle$ is said to be bounded if $\sup_{i,j,k} \bar{d}(X_{ijk}, \bar{0}) < \infty$.

A fuzzy real-valued triple sequence $X = \langle X_{ijk} \rangle$ is said to be *I*-statistically convergent to the fuzzy real number L, if for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{(i,j,k)\in N\times N\times N : \frac{1}{mnl}\left|\{(i,j,k): \bar{d}(X_{ijk},L)\ge \varepsilon, i\le m; j\le n; k\le l\}\right| \ge \delta\right\}\in I_3.$$

A fuzzy real-valued triple sequence $X=\langle X_{ijk}\rangle$ is said to be *I*-statistically pre-Cauchy if, for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{(i,j,k)\in N\times N\times N : \frac{1}{m^2n^2l^2}\left|\{(i,j,k): \bar{d}(X_{ijk},X_{pqr})\ge \varepsilon\}\right| \ge \delta, i\le m; j\le n; k\le l\right\}\in I_3.$$

3. Main Results

Theorem 3.1. If a triple sequence of fuzzy numbers $X=\langle X_{ijk}\rangle$ is *I*-statistically convergent, then $X=\langle X_{ijk}\rangle$ is *I*-statistically pre-Cauchy but an *I*-statistically pre-Cauchy triple sequence of fuzzy numbers is not necessarily *I*-statistically convergent.

Proof. Let $X=\langle X_{ijk}\rangle$ be *I*-statistically convergent to L. Then for each $\varepsilon,\delta>0$, let,

$$A=\left\{(m,n,l)\in N\times N\times N : \frac{1}{mnl}\left|\{(i,j,k): \bar{d}(X_{ijk},L)\ge \frac{\varepsilon}{2}, i\le m; j\le n; k\le l\}\right| \ge \delta\right\}\in I_3.$$

Then for $(m,n,l)\in A^c$ (complement of A), we have

$$\frac{1}{mnl}\left|\{(i,j,k): \bar{d}(X_{ijk},L)\ge \frac{\varepsilon}{2}, i\le m; j\le n; k\le l\}\right| < \delta.$$

$$\Rightarrow \frac{1}{mnl}\left|\{(i,j,k): \bar{d}(X_{ijk},L)< \frac{\varepsilon}{2}, i\le m; j\le n; k\le l\}\right| > 1-\delta.$$

Let $A_{mnl}=\{(i,j,k): \bar{d}(X_{ijk},L)<\varepsilon, i\le m; j\le n; k\le l\}$. Then $\frac{1}{mnl}|A_{mnl}|>1-\delta$ (1a)

So for all $(i,j,k)\in A_{mnl}$

$$\bar{d}(X_{ijk},X_{pqr})\le \bar{d}(X_{ijk},L)+\bar{d}(X_{pqr},L)<\varepsilon$$

$$\Rightarrow A_{mnl}\times A_{mnl}\subseteq\{(i,j,k): \bar{d}(X_{ijk},X_{pqr})<\varepsilon, i\le m; j\le n; k\le l\}.$$

$$\Rightarrow \frac{|A_{mnl}\times A_{mnl}|}{m^2n^2l^2}=\frac{[|A_{mnl}|]^2}{m^2n^2l^2}\le\frac{1}{m^2n^2l^2}\left|\{(i,j,k): \bar{d}(X_{ijk},X_{pqr})<\varepsilon, i\le m; j\le n; k\le l\}\right|$$

$$\Rightarrow \frac{1}{m^2n^2l^2}\left|\{(i,j,k): \bar{d}(X_{ijk},X_{pqr})<\varepsilon, i\le m; j\le n; k\le l\}\right|\ge\left[\frac{|A_{mnl}|}{mnl}\right]^2>(1-\delta)^2, \text{ using (1a).}$$

$$\Rightarrow \frac{1}{m^2n^2l^2}\left|\{(i,j,k): \bar{d}(X_{ijk},X_{pqr})\ge\varepsilon, i\le m; j\le n; k\le l\}\right|<1-(1-\rho)^2.$$

For any given $\delta_1>0$, let $\delta>0$, be chosen such that $1-(1-\rho)^2<\rho_1$.

Now for all $(m,n,l)\in A^c$

$$\frac{1}{m^2n^2l^2}\left|\{(i,j,k): \bar{d}(X_{ijk},X_{pqr})\ge\varepsilon, i\le m; j\le n; k\le l\}\right|<\delta_1.$$

$$\therefore\left\{(i,j,k)\in N\times N\times N : \frac{1}{m^2n^2l^2}\left|(i,j,k): \bar{d}(X_{ijk},X_{pqr})\ge\varepsilon, i\le m; j\le n; k\le l\right|\ge\delta_1\right\}\subset A\in I_3.$$

Hence $X=\langle X_{ijk}\rangle$ is *I*-statistically pre-Cauchy.

But an *I*-statistically pre-Cauchy triple sequence of fuzzy numbers is not necessarily *I*-statistically convergent, as it can be seen from the following example.

Example 3.1. Consider the fuzzy triple sequence $\langle X_{ijk}\rangle$ defined as follows:

$$X_{ijk}=\overline{A_{mnl}},\ A_{mnl}=\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\left(\frac{1}{ijk}\right),\ \text{where } (m-1)!<i<m!,(n-1)!<j<n!,(l-1)!<k<l!.$$

Then for each $0<\alpha\le 1,$ the α-cut of $\langle X_{ijk}\rangle$ is given by, $\left[X_{ijk}\right]^{\alpha}=A_{mnl}$. Now it can be easily proved that $\langle X_{ijk}\rangle$ is *I*-statistically pre-Cauchy, but not *I*-statistically convergent. ■

Theorem 3.2. Let $X=\langle X_{ijk}\rangle$ be a bounded triple sequence of fuzzy numbers. Then $X=\langle X_{ijk}\rangle$ is *I*-statistically pre-Cauchy if and only if $I-\lim\frac{1}{m^2n^2l^2}\sum_{i,p\le m}\sum_{j,q\le n}\sum_{k,r\le l}\left[\bar{d}\,(X_{ijk}\,,\,X_{pqr})\right]=0.$

$$\text{Proof. Let } I-\lim\frac{1}{m^2n^2l^2}\sum_{i,p\le m}\sum_{j,q\le n}\sum_{k,r\le l}\left[\bar{d}\,(X_{ijk}\,,\,X_{pqr})\right]=0. \tag{1b}$$

For each $\varepsilon>0,$ and $m,n,l\in N,$

$$\frac{1}{m^2n^2l^2}\sum_{i,p\le m}\sum_{j,q\le n}\sum_{k,r\le l}\left[\bar{d}\,(X_{ijk}\,,\,X_{pqr})\right]$$

$$=\frac{1}{m^2n^2l^2}\sum_{\substack{i,p\le m\\ \bar{d}(X_{nkl},X_{pqr})<\varepsilon}}\sum_{j,q\le n}\sum_{k,r\le l}\left[\bar{d}\,(X_{ijk}\,,\,X_{pqr})\right]+\frac{1}{m^2n^2l^2}\sum_{\substack{i,p\le m\\ \bar{d}(X_{nkl},X_{pqr})\ge\varepsilon}}\sum_{j,q\le n}\sum_{k,r\le l}\left[\bar{d}\,(X_{ijk}\,,\,X_{pqr})\right]$$

$$\ge\frac{1}{m^2n^2l^2}\sum_{\substack{i,p\le m\\ \bar{d}(X_{nkl},X_{pqr})\ge\varepsilon}}\sum_{j,q\le n}\sum_{k,r\le l}\left[\bar{d}\,(X_{ijk}\,,\,X_{pqr})\right]$$

$$\ge\varepsilon\left(\frac{1}{m^2n^2l^2}\left|\{(i,j,k):\bar{d}(X_{ijk},X_{pqr})\ge\varepsilon,i\le m;j\le n;k\le l\}\right|\right).$$

Therefore for any $\delta>0,$ and using (1b)

$$\left\{(m,n,l)\in N\times N\times N:\frac{1}{m^2n^2l^2}\left|\{(i,j,k):\bar{d}(X_{ijk},X_{pqr})\ge\varepsilon,i\le m;j\le n;k\le l\}\right|\ge\delta\right\}$$

$$\subset\left\{(m,n,l)\in N\times N\times N:\frac{1}{m^2n^2l^2}\sum_{i,p\le m}\sum_{j,q\le n}\sum_{k,r\le l}\left[\bar{d}\,(X_{ijk}\,,\,X_{pqr})\right]\ge\delta\varepsilon\right\}\in I_3.$$

Hence X is *I*-statistically pre-Cauchy.

Conversely let $X=\langle X_{ijk}\rangle$ be *I*-statistically pre-Cauchy. Since X is bounded, so there exists $M>0$ such that $\bar{d}(X_{ijk},\bar{0})<M$, for all $m,n,l\in N.$

The for each $\varepsilon>0,$ and $m,n,l\in N,$

$$\frac{1}{m^2n^2l^2}\sum_{i,p\le m}\sum_{j,q\le n}\sum_{k,r\le l}\sum_{k,r\le l}\left[\bar{d}\,(X_{ijk}\,,\,X_{pqr})\right]$$

$$=\frac{1}{m^2n^2l^2}\sum_{\substack{i,p\le m\\ \bar{d}(X_{nkl},L)<\frac{\varepsilon}{2}}}\sum_{j,q\le n}\sum_{k,r\le l}\left[\bar{d}\,(X_{ijk}\,,\,X_{pqr})\right]+\frac{1}{m^2n^2l^2}\sum_{\substack{i,p\le m\\ \bar{d}(X_{nkl},L)\ge\frac{\varepsilon}{2}}}\sum_{j,q\le n}\sum_{k,r\le l}\left[\bar{d}\,(X_{ijk}\,,\,X_{pqr})\right]$$

$$\le\frac{1}{m^2n^2l^2}m^2n^2l^2\frac{\varepsilon}{2}+2M\left(\frac{1}{m^2n^2l^2}\left|\{(i,j,k):\bar{d}(X_{ijk},X_{pqr})\ge\frac{\varepsilon}{2},i\le m;j\le n;k\le l\}\right|\right).$$

Since X is *I*-statistically pre-Cauchy, for any $\delta > 0$,

$$A = \left\{(m,n,l)\in N\times N\times N:\ \frac{1}{m^2n^2l^2}\left|\{(i,j,k):\bar{d}(X_{ijk},X_{pqr})\ge\frac{\varepsilon}{2}, i\le m; j\le n; k\le l\}\right|\ge\delta\right\}\in I_3.$$

Then for all $(m,n,l)\in A^c$,

$$\frac{1}{m^2n^2l^2}\left|\{(i,j,k):\bar{d}(X_{ijk},X_{pqr})\ge\frac{\varepsilon}{2}, i\le m; j\le n; k\le l\}\right|<\delta.$$

$$\therefore \frac{1}{m^2n^2l^2}\sum_{i,p\le m}\sum_{j,q\le n}\sum_{k,r\le l}\left[\bar{d}\ (X_{ijk}\ ,\ X_{pqr})\right]\le\frac{\varepsilon}{2}+2M\delta.$$

Let $\delta_1 > 0$, be chosen such that $\frac{\varepsilon}{2}+2M\delta<\delta_1$.

Then for all $(m,n,l)\in A^c$, we have

$$\frac{1}{m^2n^2l^2}\sum_{i,p\le m}\sum_{j,q\le n}\sum_{k,r\le l}\left[\bar{d}\ (X_{ijk}\ ,\ X_{pqr})\right]\le\delta_1.$$

$$\therefore\left\{(m,n,l)\in N\times N\times N:\frac{1}{m^2n^2l^2}\sum_{i,p\le m}\sum_{j,q\le n}\sum_{k,r\le l}\left[\bar{d}\ (X_{ijk}\ ,\ X_{pqr})\right]\ge\delta_1\right\}\subset A\in I_3.$$

Hence $I-\lim\frac{1}{m^2n^2l^2}\sum_{i,p\le m}\sum_{j,q\le n}\sum_{k,r\le l}\left[\bar{d}\ (X_{ijk}\ ,\ X_{pqr})\right]=0.$ ■

Theorem 3.3. Let $X=\langle X_{ijk}\rangle$ be a bounded triple sequence of fuzzy numbers. Then $X=\langle X_{ijk}\rangle$ is *I*-statistically convergent to L, if and only if $I-\lim\frac{1}{mnl}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\left[\bar{d}\ (X_{ijk}\ ,\ L)\right]=0.$

Proof. Let $I-\lim\frac{1}{mnl}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\left[\bar{d}\ (X_{ijk}\ ,\ L)\right]=0.$ (1c)

For each $\varepsilon>0$, and $m,n,l\in N$, we have

$$\frac{1}{mnl}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\left[\bar{d}\ (X_{ijk}\ ,\ L)\right]$$

$$=\frac{1}{mnl}\sum_{i=1,\bar{d}(X_{nkl},L)<\varepsilon}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\left[\bar{d}\ (X_{ijk}\ ,\ L)\right]+\frac{1}{mnl}\sum_{i=1,\bar{d}(X_{nkl},L)\ge\varepsilon}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\left[\bar{d}\ (X_{ijk}\ ,\ L)\right]\ \ge\varepsilon\left(\frac{1}{mnl}\left|\{(i,j,k):\bar{d}(X_{ijk},L)\ge\varepsilon\}\right|\right).$$

Therefore for each $\varepsilon>0$ and $\delta>0$ and using (1c),

$$\left\{(m,n,l)\in N\times N\times N:\frac{1}{mnl}\left|\{(i,j,k):\bar{d}(X_{ijk},L)\ge\varepsilon\}\right|\ge\delta\right\}$$

$$\subset\left\{(m,n,l)\in N\times N\times N:\frac{1}{mnl}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\left[\bar{d}\ (X_{ijk}\ ,\ L)\right]\ge\delta\varepsilon\right\}\in I_3.$$

Hence X is *I*-statistically convergent.

We can prove the converse part in a similar manner to the Theorem 3.2.

Theorem 3.4. Let a triple sequence of fuzzy number $X=\langle X_{ijk}\rangle$ be *I*-statistically pre-Cauchy. If X has a subsequence $\langle X_{p_{ijk}}\rangle$ which converges to L, and $0<I-\liminf\frac{1}{mnl}\left|\{(i,j,k)\in N\times N\times N: p_{ijk}\le m,n,l\}\right|<\infty$, then X is I -statistically convergent to L.

Proof. Let $\varepsilon > 0$. Since $\langle X_{p_{ijk}} \rangle$ converges to L, there exists $n_0 \in N$ such that $\bar{d}\ (X_{p_{ijk}}\ , L) < \frac{\varepsilon}{2}$ for all $p_{ijk} > n_0$.

Let $A = \{p_{ijk} : p_{ijk} > n_0; (i,j,k) \in N \times N \times N\}$ and $A(\varepsilon) = \{(i,j,k) : \bar{d}\ (X_{p_{ijk}}\ , L) \geq \varepsilon\}$.

Now $\frac{1}{m^2n^2l^2}\left|\{(i,j,k) : \bar{d}(X_{ijk}, X_{pqr}) \geq \frac{\varepsilon}{2}, i \leq m; j \leq n; k \leq l\}\right|$

$$\geq \frac{1}{m^2n^2l^2} \sum_{i,p\leq m} \sum_{j,q\leq n} \sum_{k,r\leq l} \sum_{k,r\leq l} \chi_{A(\varepsilon)\times A}(i,j,k)$$

$$= \frac{1}{mnl}\left|\{p_{ijk} \leq m,n,l.: p_{ijk} \in A\}\right| . \frac{1}{mnl}\left|\{(i,j,k) : \bar{d}(X_{ijk}, L) \geq \varepsilon; i \leq m; j \leq n; k \leq l\}\right|$$

Since X is *I*-statistically pre-Cauchy, so for $\delta > 0$,

$$B = \left\{(m,n,l) \in N \times N \times N : \frac{1}{m^2n^2l^2}\left|\{(i,j,k) : \bar{d}(X_{ijk}, X_{pqr}) \geq \frac{\varepsilon}{2}, i \leq m; j \leq n; k \leq l\}\right| \geq \delta\right\} \in I_3.$$

Thus for all $(m,n,l) \in B^C$, $\frac{1}{m^2n^2l^2}\left|\{(i,j,k) : \bar{d}(X_{ijk}, X_{pqr}) \geq \frac{\varepsilon}{2}, i \leq m; j \leq n; k \leq l\}\right| < \delta.$

Again, since, $0 < I - \liminf \frac{1}{mnl}\left|\{(i,j,k) \in N \times N \times N : p_{ijk} \leq m,n,l\}\right| = p > 0(say)$, then the set

$$C = \left\{(m,n,l) \in N \times N \times N : \frac{1}{mnl}\left|\{(i,j,k) \in N \times N \times N : p_{ijk} \leq m,n,l\}\right| < \frac{p}{2}\right\} \in I_3.$$

and so for all $(m,n,l) \in C^C$, $\frac{1}{mnl}\left|\{(i,j,k) \in N \times N \times N : p_{ijk} \leq m,n,l\}\right| \geq \frac{p}{2}.$

Therefore for all $(m,n,l) \in B^C \cap C^C = (B \cup C)^C$,

$$\frac{1}{mnl}\left|\{(i,j,k) : \bar{d}(X_{ijk}, L) \geq \frac{\varepsilon}{2}, i \leq m; j \leq n; k \leq l\}\right| < \frac{2\delta}{p}.$$

$\delta_1 > 0$, be chosen such that $\frac{2\delta}{p} < \delta_1$. Now, for all $(m,n,l) \in (B \cup C)^C$, $\frac{1}{mnl}\left|\{(i,j,k) : \bar{d}(X_{ijk}, L) \geq \frac{\varepsilon}{2}, i \leq m; j \leq n; k \leq l\}\right| < \delta_1.$

$\left\{(m,n,l) \in N \times N \times N : \frac{1}{mnl}\left|\{(i,j,k) : \bar{d}(X_{ijk}, L) \geq \varepsilon; i \leq m; j \leq n; k \leq l\}\right| \geq \delta_1\right\} \subset B \cup C \in I_3$. Hence X is *I*-statistically convergent to L.

4. Conclusion

Convergence theory is used as a basic tool in, measure spaces, sequences of random variables, information theory etc. We have introduced and studied the notion of *I*-statistically pre-Cauchy sequence of fuzzy real numbers having multiplicity greater than two. The criterion for any arbitrary triple sequence of fuzzy numbers to be *I*-statistically pre-Cauchy is derived. Also a necessary and sufficient condition for a bounded triple sequence of fuzzy real numbers to be *I*-pre-Cauchy is established.

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